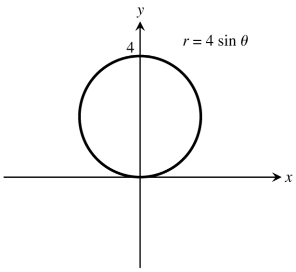
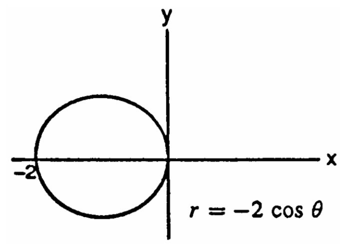
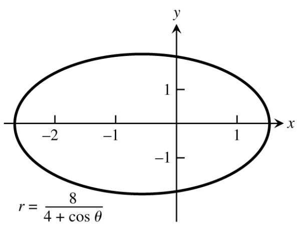
67.



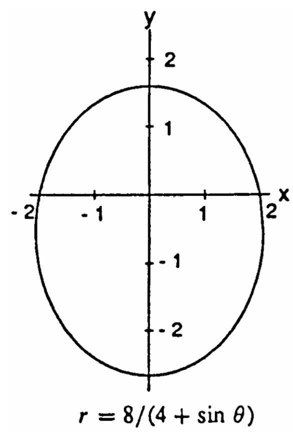
68.



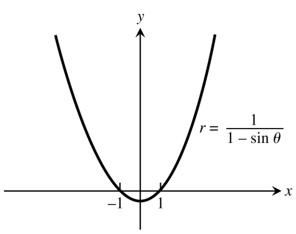
69.



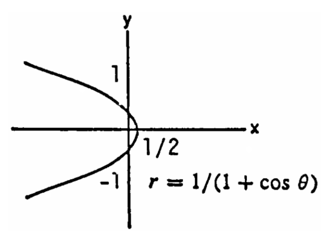
70.



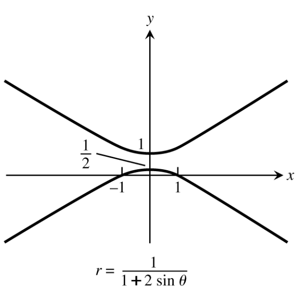
71.



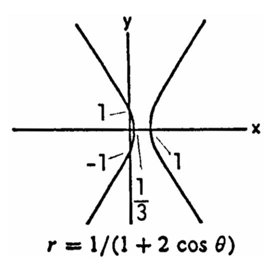
72.



73.



74.



75. (a) Perihelion $= a - ae = a(1 - e)$, Aphelion $= ea + a = a(1 + e)$

(b)

Planet	Perihelion	Aphelion
Mercury	0.3075 AU	0.4667 AU
Venus	0.7184 AU	0.7282 AU
Earth	0.9833 AU	1.0167 AU
Mars	1.3817 AU	1.6663 AU
Jupiter	4.9512 AU	5.4548 AU
Saturn	9.0210 AU	10.0570 AU
Uranus	18.2977 AU	20.0623 AU
Neptune	29.8135 AU	30.3065 AU

76. Mercury: $r = \frac{(0.3871)(1-0.2056^2)}{1+0.2056\cos\theta} = \frac{0.3707}{1+0.2056\cos\theta}$

Venus: $r = \frac{(0.7233)(1-0.0068^2)}{1+0.0068\cos\theta} = \frac{0.7233}{1+0.0068\cos\theta}$

Earth: $r = \frac{1(1-0.0167^2)}{1+0.0167\cos\theta} = \frac{0.9997}{1+0.0617\cos\theta}$

Mars: $r = \frac{(1.524)(1-0.0934^2)}{1+0.0934\cos\theta} = \frac{1.511}{1+0.0934\cos\theta}$

Jupiter: $r = \frac{(5.203)(1-0.0484^2)}{1+0.0484\cos\theta} = \frac{5.191}{1+0.0484\cos\theta}$

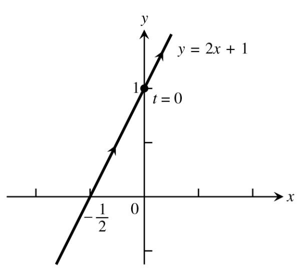
Saturn: $r = \frac{(9.539)(1-0.0543^2)}{1+0.0543\cos\theta} = \frac{9.511}{1+0.0543\cos\theta}$

Uranus: $r = \frac{(19.18)(1-0.0460^2)}{1+0.0460\cos\theta} = \frac{19.14}{1+0.0460\cos\theta}$

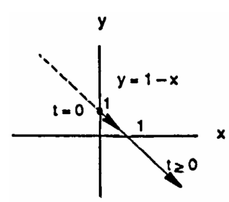
Neptune: $r = \frac{(30.06)(1-0.0082^2)}{1+0.0082\cos\theta} = \frac{30.06}{1+0.0082\cos\theta}$

CHAPTER 11 PRACTICE EXERCISES

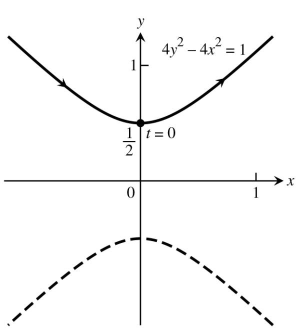
1. $x = \frac{t}{2}$ and $y = t + 1 \Rightarrow 2x = t \Rightarrow y = 2x + 1$



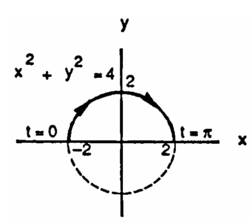
2. $x = \sqrt{t}$ and $y = 1 - \sqrt{t} \Rightarrow y = 1 - x$



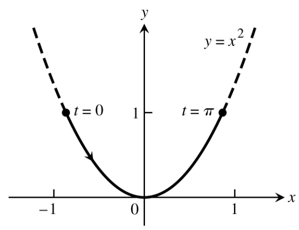
3. $x = \frac{1}{2}\tan t$ and $y = \frac{1}{2}\sec t \Rightarrow x^2 = \frac{1}{4}\tan^2 t$ and $y^2 = \frac{1}{4}\sec^2 t \Rightarrow 4x^2 = \tan^2 t$ and $4y^2 = \sec^2 t \Rightarrow 4x^2 + 1 = 4y^2 \Rightarrow 4y^2 - 4x^2 = 1$



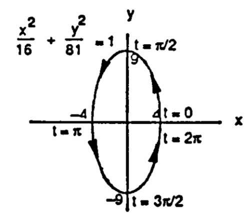
4. $x = -2\cos t$ and $y = 2\sin t \Rightarrow x^2 = 4\cos^2 t$ and $y^2 = 4\sin^2 t \Rightarrow x^2 + y^2 = 4$



5. $x = -\cos t$ and $y = \cos^2 t \Rightarrow y = (-x)^2 = x^2$



6. $x = 4 \cos t$ and $y = 9 \sin t \Rightarrow x^2 = 6 \cos^2 t$ and $y^2 = 81 \sin^2 t \Rightarrow \frac{x^2}{16} + \frac{y^2}{81} = 1$



7. $16x^2 + 9y^2 = 144 \Rightarrow \frac{x^2}{9} + \frac{y^2}{16} = 1 \Rightarrow a = 3$ and $b = 4 \Rightarrow x = 3 \cos t$ and $y = 4 \sin t$, $0 \le t \le 2\pi$

8. $x^2 + y^2 = 4 \Rightarrow x = -2 \cos t$ and $y = 2 \sin t$, $0 \le t \le 6\pi$

9. $x = \frac{1}{2} \tan t$, $y = \frac{1}{2} \sec t \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{1}{2}\sec t \tan t}{\frac{1}{2}\sec^2 t} = \frac{\tan t}{\sec t} = \sin t \Rightarrow \left.\frac{dy}{dx}\right|_{t=\pi/3} = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$; $t = \frac{\pi}{3}$
$\Rightarrow x = \frac{1}{2} \tan \frac{\pi}{3} = \frac{\sqrt{3}}{2}$ and $y = \frac{1}{2} \sec \frac{\pi}{3} = 1 \Rightarrow y = \frac{\sqrt{3}}{2} x + \frac{1}{4}$; $\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{\cos t}{\frac{1}{2}\sec^2 t} = 2 \cos^3 t \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=\pi/3}$
$= 2 \cos^3\left(\frac{\pi}{3}\right) = \frac{1}{4}$

10. $x = 1 + \frac{1}{t^2}$, $y = 1 - \frac{3}{t} \Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\left(\frac{3}{t^2}\right)}{\left(-\frac{2}{t^3}\right)} = -\frac{3}{2} t \Rightarrow \left.\frac{dy}{dx}\right|_{t=2} = -\frac{3}{2}(2) = -3$; $t = 2 \Rightarrow x = 1 + \frac{1}{2^2} = \frac{5}{4}$ and
$y = 1 - \frac{3}{2} = -\frac{1}{2} \Rightarrow y = -3x + \frac{13}{4}$; $\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{\left(-\frac{3}{2}\right)}{\left(-\frac{2}{t^3}\right)} = \frac{3}{4} t^3 \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{t=2} = \frac{3}{4}(2)^3 = 6$

11. (a) $x = 4t^2$, $y = t^3 - 1 \Rightarrow t = \pm \frac{\sqrt{x}}{2} \Rightarrow y = \left(\pm \frac{\sqrt{x}}{2}\right)^3 - 1 = \pm \frac{x^{3/2}}{8} - 1$
(b) $x = \cos t$, $y = \tan t \Rightarrow \sec t = \frac{1}{x} \Rightarrow \tan^2 t + 1 = \sec^2 t \Rightarrow y^2 = \frac{1}{x^2} - 1 = \frac{1 - x^2}{x^2} \Rightarrow y = \pm \frac{\sqrt{1 - x^2}}{x}$

12. (a) The line through $(1, -2)$ with slope 3 is $y = 3x - 5 \Rightarrow x = t$, $y = 3t - 5$, $-\infty < t < \infty$
(b) $(x-1)^2 + (y+2)^2 = 9 \Rightarrow x - 1 = 3 \cos t$, $y + 2 = 3 \sin t \Rightarrow x = 1 + 3 \cos t$, $y = -2 + 3 \sin t$, $0 \le t \le 2\pi$
(c) $y = 4x^2 - x \Rightarrow x = t$, $y = 4t^2 - t$, $-\infty < t < \infty$
(d) $9x^2 + 4y^2 = 36 \Rightarrow \frac{x^2}{4} + \frac{y^2}{9} = 1 \Rightarrow x = 2 \cos t$, $y = 3 \sin t$, $0 \le t \le 2\pi$

13. $y = x^{1/2} - \frac{x^{3/2}}{3} \Rightarrow \frac{dy}{dx} = \frac{1}{2} x^{-1/2} - \frac{1}{2} x^{1/2} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{1}{4}\left(\frac{1}{x} - 2 + x\right) \Rightarrow L = \int_1^4 \sqrt{1 + \frac{1}{4}\left(\frac{1}{x} - 2 + x\right)}\, dx$
$\Rightarrow L = \int_1^4 \sqrt{\frac{1}{4}\left(\frac{1}{x} + 2 + x\right)}\, dx = \int_1^4 \sqrt{\frac{1}{4}\left(x^{-1/2} + x^{1/2}\right)^2}\, dx = \int_1^4 \frac{1}{2}\left(x^{-1/2} + x^{1/2}\right) dx = \frac{1}{2}\left[2x^{1/2} + \frac{2}{3} x^{3/2}\right]_1^4$
$= \frac{1}{2}\left[\left(4 + \frac{2}{3} \cdot 8\right) - \left(2 + \frac{2}{3}\right)\right] = \frac{1}{2}\left(2 + \frac{14}{3}\right) = \frac{10}{3}$

14. $x = y^{2/3} \Rightarrow \frac{dx}{dy} = \frac{2}{3} x^{-1/3} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{4x^{-2/3}}{9} \Rightarrow L = \int_1^8 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_1^8 \sqrt{1 + \frac{4}{9x^{2/3}}}\, dy$
$= \int_1^8 \frac{\sqrt{9x^{2/3} + 4}}{3x^{1/3}}\, dx = \frac{1}{3} \int_1^8 \sqrt{9x^{2/3} + 4}\, \left(x^{-1/3}\right) dx$; $\left[u = 9x^{2/3} + 4 \Rightarrow du = 6y^{-1/3}\, dy; x = 1 \Rightarrow u = 13,\right.$
$\left. x = 8 \Rightarrow u = 40\right] \rightarrow L = \frac{1}{18} \int_{13}^{40} u^{1/2}\, du = \frac{1}{18}\left[\frac{2}{3} u^{3/2}\right]_{13}^{40} = \frac{1}{27}\left[40^{3/2} - 13^{3/2}\right] \approx 7.634$

15. $y = \frac{5}{12} x^{6/5} - \frac{5}{8} x^{4/5} \Rightarrow \frac{dy}{dx} = \frac{1}{2} x^{1/5} - \frac{1}{2} x^{-1/5} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{1}{4}\left(x^{2/5} - 2 + x^{-2/5}\right)$
$\Rightarrow L = \int_1^{32} \sqrt{1 + \frac{1}{4}\left(x^{2/5} - 2 + x^{-2/5}\right)}\, dx \Rightarrow L = \int_1^{32} \sqrt{\frac{1}{4}\left(x^{2/5} + 2 + x^{-2/5}\right)}\, dx = \int_1^{32} \sqrt{\frac{1}{4}\left(x^{1/5} + x^{-1/5}\right)^2}\, dx$

$= \int_1^{32} \frac{1}{2}\left(x^{1/5} + x^{-1/5}\right) dx = \frac{1}{2}\left[\frac{5}{6}x^{6/5} + \frac{5}{4}x^{4/5}\right]_1^{32} = \frac{1}{2}\left[\left(\frac{5}{6}\cdot 2^6 + \frac{5}{4}\cdot 2^4\right) - \left(\frac{5}{6} + \frac{5}{4}\right)\right] = \frac{1}{2}\left(\frac{315}{6} + \frac{75}{4}\right)$
$= \frac{1}{48}(1260 + 450) = \frac{1710}{48} = \frac{285}{8}$

16. $x = \frac{1}{12}y^3 + \frac{1}{y} \Rightarrow \frac{dx}{dy} = \frac{1}{4}y^2 - \frac{1}{y^2} \Rightarrow \left(\frac{dx}{dy}\right)^2 = \frac{1}{16}y^4 - \frac{1}{2} + \frac{1}{y^4} \Rightarrow L = \int_1^2 \sqrt{1 + \left(\frac{1}{16}y^4 - \frac{1}{2} + \frac{1}{y^4}\right)}\, dy$
$= \int_1^2 \sqrt{\frac{1}{16}y^4 + \frac{1}{2} + \frac{1}{y^4}}\, dy = \int_1^2 \sqrt{\left(\frac{1}{4}y^2 + \frac{1}{y^2}\right)^2}\, dy = \int_1^2 \left(\frac{1}{4}y^2 + \frac{1}{y^2}\right) dy = \left[\frac{1}{12}y^3 - \frac{1}{y}\right]_1^2$
$= \left(\frac{8}{12} - \frac{1}{2}\right) - \left(\frac{1}{12} - 1\right) = \frac{7}{12} + \frac{1}{2} = \frac{13}{12}$

17. $\frac{dx}{dt} = -5\sin t + 5\sin 5t$ and $\frac{dy}{dt} = 5\cos t - 5\cos 5t \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$
$= \sqrt{(-5\sin t + 5\sin 5t)^2 + (5\cos t - 5\cos 5t)^2}$
$= 5\sqrt{\sin^2 5t - 2\sin t\sin 5t + \sin^2 t + \cos^2 t - 2\cos t\cos 5t + \cos^2 5t} = 5\sqrt{2 - 2(\sin t\sin 5t + \cos t\cos 5\,t)}$
$= 5\sqrt{2(1 - \cos 4t)} = 5\sqrt{4\left(\frac{1}{2}\right)(1 - \cos 4t)} = 10\sqrt{\sin^2 2t} = 10|\sin 2t| = 10\sin 2t$ (since $0 \le t \le \frac{\pi}{2}$)
$\Rightarrow \text{Length} = \int_0^{\pi/2} 10\sin 2t\, dt = [-5\cos 2t]_0^{\pi/2} = (-5)(-1) - (-5)(1) = 10$

18. $\frac{dx}{dt} = 3t^2 - 12t$ and $\frac{dy}{dt} = 3t^2 + 12t \Rightarrow \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(3t^2 - 12t)^2 + (3t^2 + 12t)^2} = \sqrt{288t^2 + 18t^4}$
$= 3\sqrt{2}\,|t|\sqrt{16 + t^2} \Rightarrow \text{Length} = \int_0^1 3\sqrt{2}\,|t|\sqrt{16 + t^2}\, dt = 3\sqrt{2}\int_0^1 t\sqrt{16 + t^2}\, dt$; $\big[u = 16 + t^2 \Rightarrow du = 2t\, dt$
$\Rightarrow \frac{1}{2}du = t\, dt;\ t = 0 \Rightarrow u = 16;\ t = 1 \Rightarrow u = 17\big]$; $\frac{3\sqrt{2}}{2}\int_{16}^{17}\sqrt{u}\, du = \frac{3\sqrt{2}}{2}\left[\frac{2}{3}u^{3/2}\right]_{16}^{17} = \frac{3\sqrt{2}}{2}\left(\frac{2}{3}(17)^{3/2} - \frac{2}{3}(16)^{3/2}\right)$
$= \frac{3\sqrt{2}}{2}\cdot\frac{2}{3}\left((17)^{3/2} - 64\right) = \sqrt{2}\left((17)^{3/2} - 64\right) \approx 8.617.$

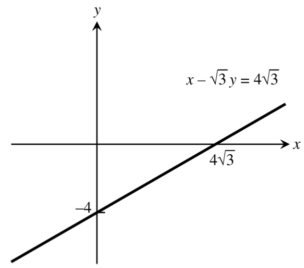
19. $\frac{dx}{d\theta} = -3\sin\theta$ and $\frac{dy}{d\theta} = 3\cos\theta \Rightarrow \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = \sqrt{(-3\sin\theta)^2 + (3\cos\theta)^2} = \sqrt{3(\sin^2\theta + \cos^2\theta)} = 3$
$\Rightarrow \text{Length} = \int_0^{3\pi/2} 3\, d\theta = 3\int_0^{3\pi/2} d\theta = 3\left(\frac{3\pi}{2} - 0\right) = \frac{9\pi}{2}$

20. $x = t^2$ and $y = \frac{t^3}{3} - t,\ -\sqrt{3} \le t \le \sqrt{3} \Rightarrow \frac{dx}{dt} = 2t$ and $\frac{dy}{dt} = t^2 - 1 \Rightarrow \text{Length} = \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{(2t)^2 + (t^2 - 1)^2}\, dt$
$= \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{t^4 + 2t^2 + 1}\, dt = \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{t^4 + 2t^2 + 1}\, dt = \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{(t^2 + 1)^2}\, dt = \int_{-\sqrt{3}}^{\sqrt{3}} (t^2 + 1)\, dt = \left[\frac{t^3}{3} + t\right]_{-\sqrt{3}}^{\sqrt{3}}$
$= 4\sqrt{3}$

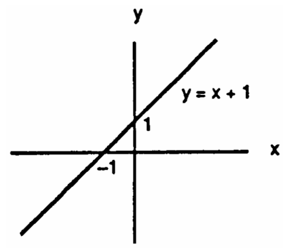
21. $x = \frac{t^2}{2}$ and $y = 2t,\ 0 \le t \le \sqrt{5} \Rightarrow \frac{dx}{dt} = t$ and $\frac{dy}{dt} = 2 \Rightarrow \text{Surface Area} = \int_0^{\sqrt{5}} 2\pi(2t)\sqrt{t^2 + 4}\, dt = \int_4^9 2\pi u^{1/2}\, du$
$= 2\pi\left[\frac{2}{3}u^{3/2}\right]_4^9 = \frac{76\pi}{3}$, where $u = t^2 + 4 \Rightarrow du = 2t\, dt;\ t = 0 \Rightarrow u = 4,\ t = \sqrt{5} \Rightarrow u = 9$

22. $x = t^2 + \frac{1}{2t}$ and $y = 4\sqrt{t},\ \frac{1}{\sqrt{2}} \le t \le 1 \Rightarrow \frac{dx}{dt} = 2t - \frac{1}{2t^2}$ and $\frac{dy}{dt} = \frac{2}{\sqrt{t}}$
$\Rightarrow \text{Surface Area} = \int_{1/\sqrt{2}}^1 2\pi\left(t^2 + \frac{1}{2t}\right)\sqrt{\left(2t - \frac{1}{2t^2}\right)^2 + \left(\frac{2}{\sqrt{t}}\right)^2}\, dt = 2\pi\int_{1/\sqrt{2}}^1 \left(t^2 + \frac{1}{2t}\right)\sqrt{\left(2t + \frac{1}{2t^2}\right)^2}\, dt$
$= 2\pi\int_{1/\sqrt{2}}^1 \left(t^2 + \frac{1}{2t}\right)\left(2t + \frac{1}{2t^2}\right) dt = 2\pi\int_{1/\sqrt{2}}^1 \left(2t^3 + \frac{3}{2} + \frac{1}{4}t^{-3}\right) dt = 2\pi\left[\frac{1}{2}t^4 + \frac{3}{2}t - \frac{1}{8}t^{-2}\right]_{1/\sqrt{2}}^1$
$= 2\pi\left(2 - \frac{3\sqrt{2}}{4}\right)$

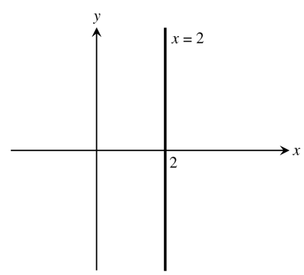
23. $r\cos\left(\theta+\frac{\pi}{3}\right)=2\sqrt{3} \Rightarrow r\left(\cos\theta\cos\frac{\pi}{3}-\sin\theta\sin\frac{\pi}{3}\right)$
$=2\sqrt{3} \Rightarrow \frac{1}{2}r\cos\theta-\frac{\sqrt{3}}{2}r\sin\theta=2\sqrt{3}$
$\Rightarrow r\cos\theta-\sqrt{3}r\sin\theta=4\sqrt{3} \Rightarrow x-\sqrt{3}y=4\sqrt{3}$
$\Rightarrow y=\frac{\sqrt{3}}{3}x-4$



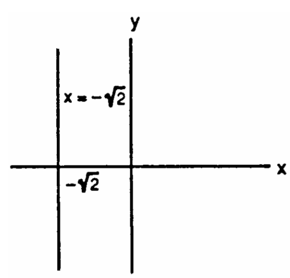
24. $r\cos\left(\theta-\frac{3\pi}{4}\right)=\frac{\sqrt{2}}{2} \Rightarrow r\left(\cos\theta\cos\frac{3\pi}{4}+\sin\theta\sin\frac{3\pi}{4}\right)$
$=\frac{\sqrt{2}}{2} \Rightarrow -\frac{\sqrt{2}}{2}r\cos\theta+\frac{\sqrt{2}}{2}r\sin\theta=\frac{\sqrt{2}}{2} \Rightarrow -x+y=1$
$\Rightarrow y=x+1$



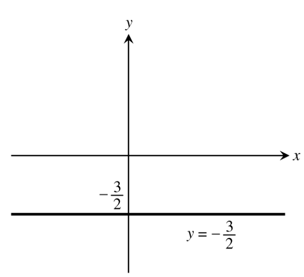
25. $r=2\sec\theta \Rightarrow r=\frac{2}{\cos\theta} \Rightarrow r\cos\theta=2 \Rightarrow x=2$



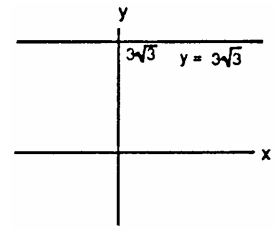
26. $r=-\sqrt{2}\sec\theta \Rightarrow r\cos\theta=-\sqrt{2} \Rightarrow x=-\sqrt{2}$



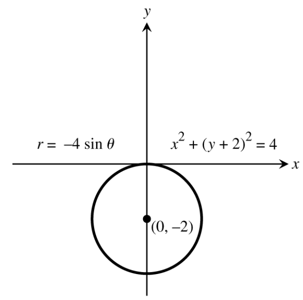
27. $r=-\frac{3}{2}\csc\theta \Rightarrow r\sin\theta=-\frac{3}{2} \Rightarrow y=-\frac{3}{2}$



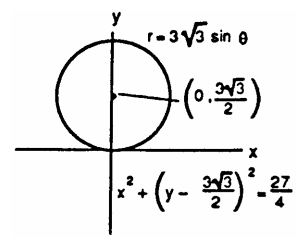
28. $r = 3\sqrt{3}\csc\theta \Rightarrow r\sin\theta = 3\sqrt{3} \Rightarrow y = 3\sqrt{3}$



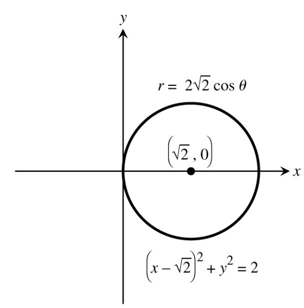
29. $r = -4\sin\theta \Rightarrow r^2 = -4r\sin\theta \Rightarrow x^2 + y^2 + 4y = 0$
$\Rightarrow x^2 + (y+2)^2 = 4$; circle with center $(0, -2)$ and radius 2.



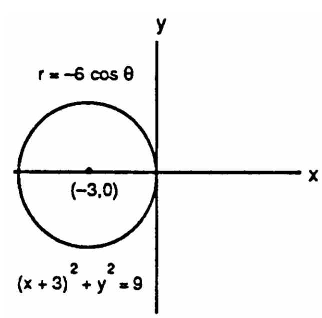
30. $r = 3\sqrt{3}\sin\theta \Rightarrow r^2 = 3\sqrt{3}\,r\sin\theta$
$\Rightarrow x^2 + y^2 - 3\sqrt{3}\,y = 0 \Rightarrow x^2 + \left(y - \frac{3\sqrt{3}}{2}\right)^2 = \frac{27}{4}$;
circle with center $\left(0, \frac{3\sqrt{3}}{2}\right)$ and radius $\frac{3\sqrt{3}}{2}$



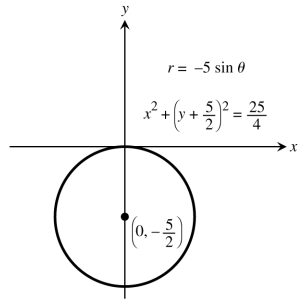
31. $r = 2\sqrt{2}\cos\theta \Rightarrow r^2 = 2\sqrt{2}\,r\cos\theta$
$\Rightarrow x^2 + y^2 - 2\sqrt{2}\,x = 0 \Rightarrow \left(x - \sqrt{2}\right)^2 + y^2 = 2$;
circle with center $\left(\sqrt{2}, 0\right)$ and radius $\sqrt{2}$



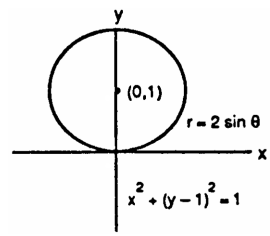
32. $r = -6\cos\theta \Rightarrow r^2 = -6r\cos\theta \Rightarrow x^2 + y^2 + 6x = 0$
$\Rightarrow (x+3)^2 + y^2 = 9$; circle with center $(-3, 0)$ and radius 3



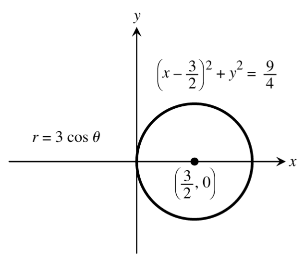
33. $x^2 + y^2 + 5y = 0 \Rightarrow x^2 + \left(y + \frac{5}{2}\right)^2 = \frac{25}{4} \Rightarrow C = \left(0, -\frac{5}{2}\right)$ and $a = \frac{5}{2}$; $r^2 + 5r \sin\theta = 0 \Rightarrow r = -5 \sin\theta$



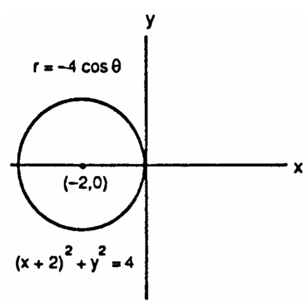
34. $x^2 + y^2 - 2y = 0 \Rightarrow x^2 + (y-1)^2 = 1 \Rightarrow C = (0, 1)$ and $a = 1$; $r^2 - 2r \sin\theta = 0 \Rightarrow r = 2 \sin\theta$



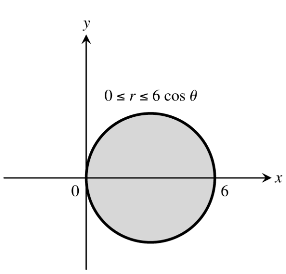
35. $x^2 + y^2 - 3x = 0 \Rightarrow \left(x - \frac{3}{2}\right)^2 + y^2 = \frac{9}{4} \Rightarrow C = \left(\frac{3}{2}, 0\right)$ and $a = \frac{3}{2}$; $r^2 - 3r \cos\theta = 0 \Rightarrow r = 3 \cos\theta$



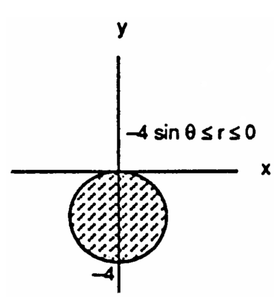
36. $x^2 + y^2 + 4x = 0 \Rightarrow (x+2)^2 + y^2 = 4 \Rightarrow C = (-2, 0)$ and $a = 2$; $r^2 + 4r \cos\theta = 0 \Rightarrow r = -4 \cos\theta$



37.



38.



39. d
40. e
41. l
42. f
43. k
44. h
45. i
46. j

47. $A = 2\int_0^\pi \frac{1}{2}\, r^2\, d\theta = \int_0^\pi (2 - \cos\theta)^2\, d\theta = \int_0^\pi (4 - 4\cos\theta + \cos^2\theta)\, d\theta = \int_0^\pi \left(4 - 4\cos\theta + \frac{1 + \cos 2\theta}{2}\right) d\theta$
$= \int_0^\pi \left(\frac{9}{2} - 4\cos\theta + \frac{\cos 2\theta}{2}\right) d\theta = \left[\frac{9}{2}\theta - 4\sin\theta + \frac{\sin 2\theta}{4}\right]_0^\pi = \frac{9}{2}\pi$

48. $A = \int_0^{\pi/3} \frac{1}{2}\,(\sin^2 3\theta)\, d\theta = \int_0^{\pi/3} \left(\frac{1 - \cos 6\theta}{2}\right) d\theta = \frac{1}{4}\left[\theta - \frac{1}{6}\sin 6\theta\right]_0^{\pi/3} = \frac{\pi}{12}$

49. $r = 1 + \cos 2\theta$ and $r = 1 \Rightarrow 1 = 1 + \cos 2\theta \Rightarrow 0 = \cos 2\theta \Rightarrow 2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$; therefore
$A = 4\int_0^{\pi/4} \frac{1}{2}\left[(1 + \cos 2\theta)^2 - 1^2\right] d\theta = 2\int_0^{\pi/4} (1 + 2\cos 2\theta + \cos^2 2\theta - 1)\, d\theta$
$= 2\int_0^{\pi/4} \left(2\cos 2\theta + \frac{1}{2} + \frac{\cos 4\theta}{2}\right) d\theta = 2\left[\sin 2\theta + \frac{1}{2}\theta + \frac{\sin 4\theta}{8}\right]_0^{\pi/4} = 2\left(1 + \frac{\pi}{8} + 0\right) = 2 + \frac{\pi}{4}$

50. The circle lies interior to the cardioid. Thus,
$A = 2\int_{-\pi/2}^{\pi/2} \frac{1}{2}\left[2(1 + \sin\theta)\right]^2 d\theta - \pi$ (the integral is the area of the cardioid minus the area of the circle)
$= \int_{-\pi/2}^{\pi/2} 4\,(1 + 2\sin\theta + \sin^2\theta)\, d\theta - \pi = \int_{-\pi/2}^{\pi/2} (6 + 8\sin\theta - 2\cos 2\theta)\, d\theta - \pi = \left[6\theta - 8\cos\theta - \sin 2\theta\right]_{-\pi/2}^{\pi/2} - \pi$
$= [3\pi - (-3\pi)] - \pi = 5\pi$

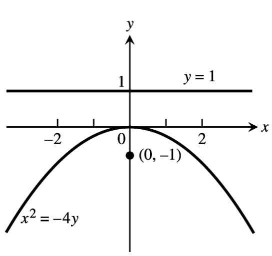
51. $r = -1 + \cos\theta \Rightarrow \frac{dr}{d\theta} = -\sin\theta$; Length $= \int_0^{2\pi} \sqrt{(-1 + \cos\theta)^2 + (-\sin\theta)^2}\, d\theta = \int_0^{2\pi} \sqrt{2 - 2\cos\theta}\, d\theta$
$= \int_0^{2\pi} \sqrt{\frac{4(1 - \cos\theta)}{2}}\, d\theta = \int_0^{2\pi} 2\sin\frac{\theta}{2}\, d\theta = \left[-4\cos\frac{\theta}{2}\right]_0^{2\pi} = (-4)(-1) - (-4)(1) = 8$

52. $r = 2\sin\theta + 2\cos\theta,\ 0 \le \theta \le \frac{\pi}{2} \Rightarrow \frac{dr}{d\theta} = 2\cos\theta - 2\sin\theta;\ r^2 + \left(\frac{dr}{d\theta}\right)^2 = (2\sin\theta + 2\cos\theta)^2 + (2\cos\theta - 2\sin\theta)^2$
$= 8\,(\sin^2\theta + \cos^2\theta) = 8 \Rightarrow L = \int_0^{\pi/2} \sqrt{8}\, d\theta = \left[2\sqrt{2}\,\theta\right]_0^{\pi/2} = 2\sqrt{2}\left(\frac{\pi}{2}\right) = \pi\sqrt{2}$

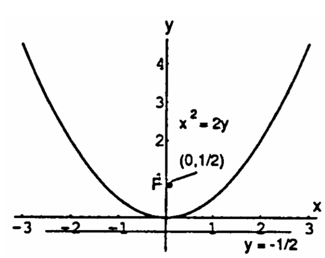
53. $r = 8\sin^3\left(\frac{\theta}{3}\right),\ 0 \le \theta \le \frac{\pi}{4} \Rightarrow \frac{dr}{d\theta} = 8\sin^2\left(\frac{\theta}{3}\right)\cos\left(\frac{\theta}{3}\right);\ r^2 + \left(\frac{dr}{d\theta}\right)^2 = \left[8\sin^3\left(\frac{\theta}{3}\right)\right]^2 + \left[8\sin^2\left(\frac{\theta}{3}\right)\cos\left(\frac{\theta}{3}\right)\right]^2$
$= 64\sin^4\left(\frac{\theta}{3}\right) \Rightarrow L = \int_0^{\pi/4} \sqrt{64\sin^4\left(\frac{\theta}{3}\right)}\, d\theta = \int_0^{\pi/4} 8\sin^2\left(\frac{\theta}{3}\right) d\theta = \int_0^{\pi/4} 8\left[\frac{1 - \cos\left(\frac{2\theta}{3}\right)}{2}\right] d\theta$
$= \int_0^{\pi/4} \left[4 - 4\cos\left(\frac{2\theta}{3}\right)\right] d\theta = \left[4\theta - 6\sin\left(\frac{2\theta}{3}\right)\right]_0^{\pi/4} = 4\left(\frac{\pi}{4}\right) - 6\sin\left(\frac{\pi}{6}\right) - 0 = \pi - 3$

54. $r = \sqrt{1 + \cos 2\theta} \Rightarrow \frac{dr}{d\theta} = \frac{1}{2}(1 + \cos 2\theta)^{-1/2}(-2\sin 2\theta) = \frac{-\sin 2\theta}{\sqrt{1 + \cos 2\theta}} \Rightarrow \left(\frac{dr}{d\theta}\right)^2 = \frac{\sin^2 2\theta}{1 + \cos 2\theta}$
$\Rightarrow r^2 + \left(\frac{dr}{d\theta}\right)^2 = 1 + \cos 2\theta + \frac{\sin^2 2\theta}{1 + \cos 2\theta} = \frac{(1 + \cos 2\theta)^2 + \sin^2 2\theta}{1 + \cos 2\theta} = \frac{1 + 2\cos 2\theta + \cos^2 2\theta + \sin^2 2\theta}{1 + \cos 2\theta}$
$= \frac{2 + 2\cos 2\theta}{1 + \cos 2\theta} = 2 \Rightarrow L = \int_{-\pi/2}^{\pi/2} \sqrt{2}\, d\theta = \sqrt{2}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = \sqrt{2}\,\pi$

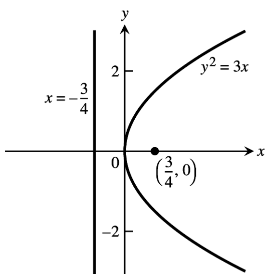
55. $x^2 = -4y \Rightarrow y = -\frac{x^2}{4} \Rightarrow 4p = 4 \Rightarrow p = 1$;
therefore Focus is $(0, -1)$, Directrix is $y = 1$



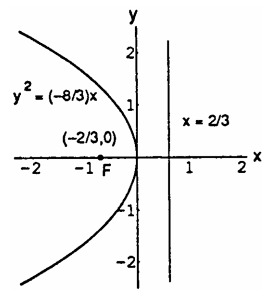
56. $x^2 = 2y \Rightarrow \frac{x^2}{2} = y \Rightarrow 4p = 2 \Rightarrow p = \frac{1}{2}$;
therefore Focus is $\left(0, \frac{1}{2}\right)$; Directrix is $y = -\frac{1}{2}$



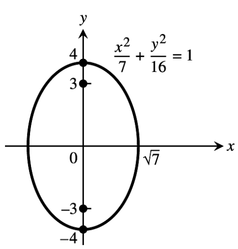
57. $y^2 = 3x \Rightarrow x = \frac{y^2}{3} \Rightarrow 4p = 3 \Rightarrow p = \frac{3}{4}$; therefore Focus is $\left(\frac{3}{4}, 0\right)$, Directrix is $x = -\frac{3}{4}$



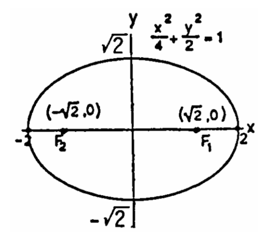
58. $y^2 = -\frac{8}{3}x \Rightarrow x = -\frac{y^2}{\left(\frac{8}{3}\right)} \Rightarrow 4p = \frac{8}{3} \Rightarrow p = \frac{2}{3}$; therefore Focus is $\left(-\frac{2}{3}, 0\right)$, Directrix is $x = \frac{2}{3}$



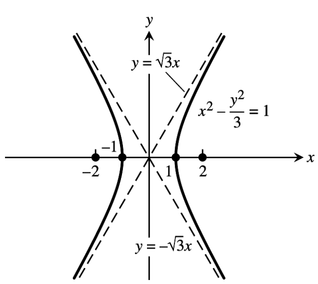
59. $16x^2 + 7y^2 = 112 \Rightarrow \frac{x^2}{7} + \frac{y^2}{16} = 1$ $\Rightarrow c^2 = 16 - 7 = 9 \Rightarrow c = 3; e = \frac{c}{a} = \frac{3}{4}$



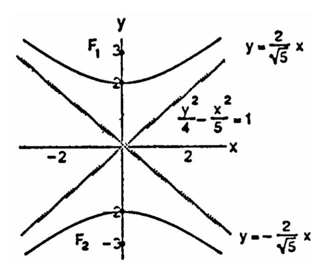
60. $x^2 + 2y^2 = 4 \Rightarrow \frac{x^2}{4} + \frac{y^2}{2} = 1 \Rightarrow c^2 = 4 - 2 = 2$ $\Rightarrow c = \sqrt{2}; e = \frac{c}{a} = \frac{\sqrt{2}}{2}$



61. $3x^2 - y^2 = 3 \Rightarrow x^2 - \frac{y^2}{3} = 1 \Rightarrow c^2 = 1 + 3 = 4$ $\Rightarrow c = 2; e = \frac{c}{a} = \frac{2}{1} = 2$; the asymptotes are $y = \pm\sqrt{3}\,x$



62. $5y^2 - 4x^2 = 20 \Rightarrow \frac{y^2}{4} - \frac{x^2}{5} = 1 \Rightarrow c^2 = 4 + 5 = 9$ $\Rightarrow c = 3, e = \frac{c}{a} = \frac{3}{2}$; the asymptotes are $y = \pm\frac{2}{\sqrt{5}}\,x$



63. $x^2 = -12y \Rightarrow -\frac{x^2}{12} = y \Rightarrow 4p = 12 \Rightarrow p = 3 \Rightarrow$ focus is $(0, -3)$, directrix is $y = 3$, vertex is $(0, 0)$; therefore new vertex is $(2, 3)$, new focus is $(2, 0)$, new directrix is $y = 6$, and the new equation is $(x - 2)^2 = -12(y - 3)$

64. $y^2 = 10x \Rightarrow \frac{y^2}{10} = x \Rightarrow 4p = 10 \Rightarrow p = \frac{5}{2} \Rightarrow$ focus is $\left(\frac{5}{2}, 0\right)$, directrix is $x = -\frac{5}{2}$, vertex is $(0, 0)$; therefore new vertex is $\left(-\frac{1}{2}, -1\right)$, new focus is $(2, -1)$, new directrix is $x = -3$, and the new equation is $(y + 1)^2 = 10\left(x + \frac{1}{2}\right)$

65. $\frac{x^2}{9} + \frac{y^2}{25} = 1 \Rightarrow a = 5$ and $b = 3 \Rightarrow c = \sqrt{25 - 9} = 4 \Rightarrow$ foci are $(0, \pm 4)$, vertices are $(0, \pm 5)$, center is $(0, 0)$; therefore the new center is $(-3, -5)$, new foci are $(-3, -1)$ and $(-3, -9)$, new vertices are $(-3, -10)$ and $(-3, 0)$, and the new equation is $\frac{(x+3)^2}{9} + \frac{(y+5)^2}{25} = 1$

66. $\frac{x^2}{169} + \frac{y^2}{144} = 1 \Rightarrow a = 13$ and $b = 12 \Rightarrow c = \sqrt{169 - 144} = 5 \Rightarrow$ foci are $(\pm 5, 0)$, vertices are $(\pm 13, 0)$, center is $(0, 0)$; therefore the new center is $(5, 12)$, new foci are $(10, 12)$ and $(0, 12)$, new vertices are $(18, 12)$ and $(-8, 12)$, and the new equation is $\frac{(x-5)^2}{169} + \frac{(y-12)^2}{144} = 1$

67. $\frac{y^2}{8} - \frac{x^2}{2} = 1 \Rightarrow a = 2\sqrt{2}$ and $b = \sqrt{2} \Rightarrow c = \sqrt{8+2} = \sqrt{10} \Rightarrow$ foci are $\left(0, \pm\sqrt{10}\right)$, vertices are $\left(0, \pm 2\sqrt{2}\right)$, center is $(0, 0)$, and the asymptotes are $y = \pm 2x$; therefore the new center is $\left(2, 2\sqrt{2}\right)$, new foci are $\left(2, 2\sqrt{2} \pm \sqrt{10}\right)$, new vertices are $\left(2, 4\sqrt{2}\right)$ and $(2, 0)$, the new asymptotes are $y = 2x - 4 + 2\sqrt{2}$ and $y = -2x + 4 + 2\sqrt{2}$; the new equation is $\frac{\left(y - 2\sqrt{2}\right)^2}{8} - \frac{(x-2)^2}{2} = 1$

68. $\frac{x^2}{36} - \frac{y^2}{64} = 1 \Rightarrow a = 6$ and $b = 8 \Rightarrow c = \sqrt{36 + 64} = 10 \Rightarrow$ foci are $(\pm 10, 0)$, vertices are $(\pm 6, 0)$, the center is $(0, 0)$ and the asymptotes are $\frac{y}{8} = \pm\frac{x}{6}$ or $y = \pm\frac{4}{3}x$; therefore the new center is $(-10, -3)$, the new foci are $(-20, -3)$ and $(0, -3)$, the new vertices are $(-16, -3)$ and $(-4, -3)$, the new asymptotes are $y = \frac{4}{3}x + \frac{31}{3}$ and $y = -\frac{4}{3}x - \frac{49}{3}$; the new equation is $\frac{(x+10)^2}{36} - \frac{(y+3)^2}{64} = 1$

69. $x^2 - 4x - 4y^2 = 0 \Rightarrow x^2 - 4x + 4 - 4y^2 = 4 \Rightarrow (x-2)^2 - 4y^2 = 4 \Rightarrow \frac{(x-2)^2}{4} - y^2 = 1$, a hyperbola; $a = 2$ and $b = 1 \Rightarrow c = \sqrt{1+4} = \sqrt{5}$; the center is $(2, 0)$, the vertices are $(0, 0)$ and $(4, 0)$; the foci are $\left(2 \pm \sqrt{5}, 0\right)$ and the asymptotes are $y = \pm\frac{x-2}{2}$

70. $4x^2 - y^2 + 4y = 8 \Rightarrow 4x^2 - y^2 + 4y - 4 = 4 \Rightarrow 4x^2 - (y-2)^2 = 4 \Rightarrow x^2 - \frac{(y-2)^2}{4} = 1$, a hyperbola; $a = 1$ and $b = 2 \Rightarrow c = \sqrt{1+4} = \sqrt{5}$; the center is $(0, 2)$, the vertices are $(1, 2)$ and $(-1, 2)$, the foci are $\left(\pm\sqrt{5}, 2\right)$ and the asymptotes are $y = \pm 2x + 2$

71. $y^2 - 2y + 16x = -49 \Rightarrow y^2 - 2y + 1 = -16x - 48 \Rightarrow (y-1)^2 = -16(x+3)$, a parabola; the vertex is $(-3, 1)$; $4p = 16 \Rightarrow p = 4 \Rightarrow$ the focus is $(-7, 1)$ and the directrix is $x = 1$

72. $x^2 - 2x + 8y = -17 \Rightarrow x^2 - 2x + 1 = -8y - 16 \Rightarrow (x-1)^2 = -8(y+2)$, a parabola; the vertex is $(1, -2)$; $4p = 8 \Rightarrow p = 2 \Rightarrow$ the focus is $(1, -4)$ and the directrix is $y = 0$

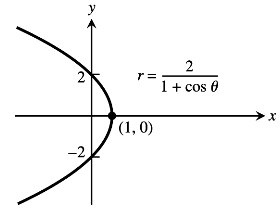
73. $9x^2 + 16y^2 + 54x - 64y = -1 \Rightarrow 9(x^2 + 6x) + 16(y^2 - 4y) = -1 \Rightarrow 9(x^2 + 6x + 9) + 16(y^2 - 4y + 4) = 144$ $\Rightarrow 9(x+3)^2 + 16(y-2)^2 = 144 \Rightarrow \frac{(x+3)^2}{16} + \frac{(y-2)^2}{9} = 1$, an ellipse; the center is $(-3, 2)$; $a = 4$ and $b = 3$ $\Rightarrow c = \sqrt{16 - 9} = \sqrt{7}$; the foci are $\left(-3 \pm \sqrt{7}, 2\right)$; the vertices are $(1, 2)$ and $(-7, 2)$

74. $25x^2 + 9y^2 - 100x + 54y = 44 \Rightarrow 25(x^2 - 4x) + 9(y^2 + 6y) = 44 \Rightarrow 25(x^2 - 4x + 4) + 9(y^2 + 6y + 9) = 225$ $\Rightarrow \frac{(x-2)^2}{9} + \frac{(y+3)^2}{25} = 1$, an ellipse; the center is $(2, -3)$; $a = 5$ and $b = 3 \Rightarrow c = \sqrt{25 - 9} = 4$; the foci are $(2, 1)$ and $(2, -7)$; the vertices are $(2, 2)$ and $(2, -8)$

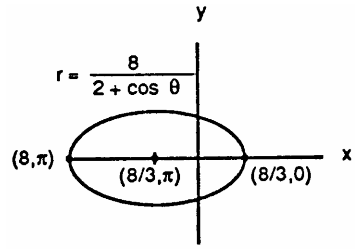
75. $x^2 + y^2 - 2x - 2y = 0 \Rightarrow x^2 - 2x + 1 + y^2 - 2y + 1 = 2 \Rightarrow (x-1)^2 + (y-1)^2 = 2$, a circle with center $(1, 1)$ and radius $= \sqrt{2}$

76. $x^2 + y^2 + 4x + 2y = 1 \Rightarrow x^2 + 4x + 4 + y^2 + 2y + 1 = 6 \Rightarrow (x+2)^2 + (y+1)^2 = 6$, a circle with center $(-2, -1)$ and radius $= \sqrt{6}$

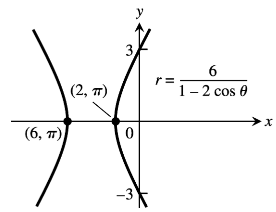
77. $r = \frac{2}{1+\cos\theta} \Rightarrow e = 1 \Rightarrow$ parabola with vertex at $(1, 0)$



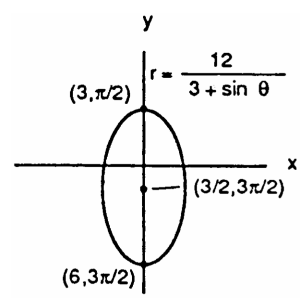
78. $r = \frac{8}{2+\cos\theta} \Rightarrow r = \frac{4}{1+\left(\frac{1}{2}\right)\cos\theta} \Rightarrow e = \frac{1}{2} \Rightarrow$ ellipse; $ke = 4 \Rightarrow \frac{1}{2}k = 4 \Rightarrow k = 8$; $k = \frac{a}{e} - ea \Rightarrow 8 = \frac{a}{\left(\frac{1}{2}\right)} - \frac{1}{2}a$ $\Rightarrow a = \frac{16}{3} \Rightarrow ea = \left(\frac{1}{2}\right)\left(\frac{16}{3}\right) = \frac{8}{3}$; therefore the center is $\left(\frac{8}{3}, \pi\right)$; vertices are $(8, \pi)$ and $\left(\frac{8}{3}, 0\right)$



79. $r = \frac{6}{1-2\cos\theta} \Rightarrow e = 2 \Rightarrow$ hyperbola; $ke = 6 \Rightarrow 2k = 6$ $\Rightarrow k = 3 \Rightarrow$ vertices are $(2, \pi)$ and $(6, \pi)$



80. $r = \frac{12}{3+\sin\theta} \Rightarrow r = \frac{4}{1+\left(\frac{1}{3}\right)\sin\theta} \Rightarrow e = \frac{1}{3}$; $ke = 4$ $\Rightarrow \frac{1}{3}k = 4 \Rightarrow k = 12$; $a\left(1 - e^2\right) = 4 \Rightarrow a\left[1 - \left(\frac{1}{3}\right)^2\right]$ $= 4 \Rightarrow a = \frac{9}{2} \Rightarrow ea = \left(\frac{1}{3}\right)\left(\frac{9}{2}\right) = \frac{3}{2}$; therefore the center is $\left(\frac{3}{2}, \frac{3\pi}{2}\right)$; vertices are $\left(3, \frac{\pi}{2}\right)$ and $\left(6, \frac{3\pi}{2}\right)$



81. $e = 2$ and $r\cos\theta = 2 \Rightarrow x = 2$ is directrix $\Rightarrow k = 2$; the conic is a hyperbola; $r = \frac{ke}{1+e\cos\theta} \Rightarrow r = \frac{(2)(2)}{1+2\cos\theta}$ $\Rightarrow r = \frac{4}{1+2\cos\theta}$

82. $e = 1$ and $r\cos\theta = -4 \Rightarrow x = -4$ is directrix $\Rightarrow k = 4$; the conic is a parabola; $r = \frac{ke}{1-e\cos\theta} \Rightarrow r = \frac{(4)(1)}{1-\cos\theta}$ $\Rightarrow r = \frac{4}{1-\cos\theta}$

83. $e = \frac{1}{2}$ and $r\sin\theta = 2 \Rightarrow y = 2$ is directrix $\Rightarrow k = 2$; the conic is an ellipse; $r = \frac{ke}{1+e\sin\theta} \Rightarrow r = \frac{(2)\left(\frac{1}{2}\right)}{1+\left(\frac{1}{2}\right)\sin\theta}$ $\Rightarrow r = \frac{2}{2+\sin\theta}$

84. $e = \frac{1}{3}$ and $r\sin\theta = -6 \Rightarrow y = -6$ is directrix $\Rightarrow k = 6$; the conic is an ellipse; $r = \frac{ke}{1-e\sin\theta} \Rightarrow r = \frac{(6)\left(\frac{1}{3}\right)}{1-\left(\frac{1}{3}\right)\sin\theta}$ $\Rightarrow r = \frac{6}{3-\sin\theta}$

85. (a) Around the x-axis: $9x^2 + 4y^2 = 36 \Rightarrow y^2 = 9 - \frac{9}{4}x^2 \Rightarrow y = \pm\sqrt{9 - \frac{9}{4}x^2}$ and we use the positive root:

$$V = 2\int_0^2 \pi\left(\sqrt{9 - \frac{9}{4}x^2}\right)^2 dx = 2\int_0^2 \pi\left(9 - \frac{9}{4}x^2\right) dx = 2\pi\left[9x - \frac{3}{4}x^3\right]_0^2 = 24\pi$$

(b) Around the y-axis: $9x^2 + 4y^2 = 36 \Rightarrow x^2 = 4 - \frac{4}{9}y^2 \Rightarrow x = \pm\sqrt{4 - \frac{4}{9}y^2}$ and we use the positive root:
$V = 2\int_0^3 \pi\left(\sqrt{4 - \frac{4}{9}y^2}\right)^2 dy = 2\int_0^3 \pi\left(4 - \frac{4}{9}y^2\right) dy = 2\pi\left[4y - \frac{4}{27}y^3\right]_0^3 = 16\pi$

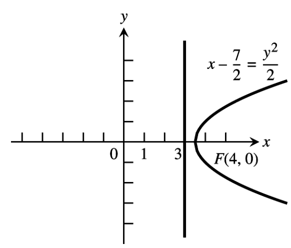
86. $9x^2 - 4y^2 = 36,\ x = 4 \Rightarrow y^2 = \frac{9x^2 - 36}{4} \Rightarrow y = \frac{3}{2}\sqrt{x^2 - 4}$; $V = \int_2^4 \pi\left(\frac{3}{2}\sqrt{x^2 - 4}\right)^2 dx = \frac{9\pi}{4}\int_2^4 (x^2 - 4)\, dx$
$= \frac{9\pi}{4}\left[\frac{x^3}{3} - 4x\right]_2^4 = \frac{9\pi}{4}\left[\left(\frac{64}{3} - 16\right) - \left(\frac{8}{3} - 8\right)\right] = \frac{9\pi}{4}\left(\frac{56}{3} - \frac{24}{3}\right) = \frac{3\pi}{4}(32) = 24\pi$

87. (a) $r = \frac{k}{1 + e\cos\theta} \Rightarrow r + er\cos\theta = k \Rightarrow \sqrt{x^2 + y^2} + ex = k \Rightarrow \sqrt{x^2 + y^2} = k - ex \Rightarrow x^2 + y^2$
$= k^2 - 2kex + e^2x^2 \Rightarrow x^2 - e^2x^2 + y^2 + 2kex - k^2 = 0 \Rightarrow (1 - e^2)x^2 + y^2 + 2kex - k^2 = 0$
(b) $e = 0 \Rightarrow x^2 + y^2 - k^2 = 0 \Rightarrow x^2 + y^2 = k^2 \Rightarrow$ circle;
$0 < e < 1 \Rightarrow e^2 < 1 \Rightarrow e^2 - 1 < 0 \Rightarrow B^2 - 4AC = 0^2 - 4(1 - e^2)(1) = 4(e^2 - 1) < 0 \Rightarrow$ ellipse;
$e = 1 \Rightarrow B^2 - 4AC = 0^2 - 4(0)(1) = 0 \Rightarrow$ parabola;
$e > 1 \Rightarrow e^2 > 1 \Rightarrow B^2 - 4AC = 0^2 - 4(1 - e^2)(1) = 4e^2 - 4 > 0 \Rightarrow$ hyperbola

88. Let (r_1, θ_1) be a point on the graph where $r_1 = a\theta_1$. Let (r_2, θ_2) be on the graph where $r_2 = a\theta_2$ and $\theta_2 = \theta_1 + 2\pi$. Then r_1 and r_2 lie on the same ray on consecutive turns of the spiral and the distance between the two points is $r_2 - r_1 = a\theta_2 - a\theta_1 = a(\theta_2 - \theta_1) = 2\pi a$, which is constant.

CHAPTER 11 ADDITIONAL AND ADVANCED EXERCISES

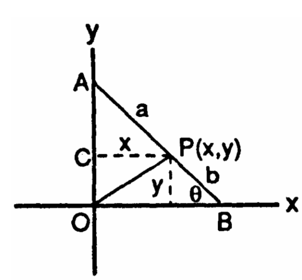
1. Directrix $x = 3$ and focus $(4, 0) \Rightarrow$ vertex is $\left(\frac{7}{2}, 0\right)$
$\Rightarrow p = \frac{1}{2} \Rightarrow$ the equation is $x - \frac{7}{2} = \frac{y^2}{2}$



2. $x^2 - 6x - 12y + 9 = 0 \Rightarrow x^2 - 6x + 9 = 12y \Rightarrow \frac{(x-3)^2}{12} = y \Rightarrow$ vertex is $(3, 0)$ and $p = 3 \Rightarrow$ focus is $(3, 3)$ and the directrix is $y = -3$

3. $x^2 = 4y \Rightarrow$ vertex is $(0, 0)$ and $p = 1 \Rightarrow$ focus is $(0, 1)$; thus the distance from $P(x, y)$ to the vertex is $\sqrt{x^2 + y^2}$ and the distance from P to the focus is $\sqrt{x^2 + (y - 1)^2} \Rightarrow \sqrt{x^2 + y^2} = 2\sqrt{x^2 + (y - 1)^2}$
$\Rightarrow x^2 + y^2 = 4\left[x^2 + (y - 1)^2\right] \Rightarrow x^2 + y^2 = 4x^2 + 4y^2 - 8y + 4 \Rightarrow 3x^2 + 3y^2 - 8y + 4 = 0$, which is a circle

4. Let the segment $a + b$ intersect the y-axis in point A and intersect the x-axis in point B so that $PB = b$ and $PA = a$ (see figure). Draw the horizontal line through P and let it intersect the y-axis in point C. Let $\angle PBO = \theta$
$\Rightarrow \angle APC = \theta$. Then $\sin\theta = \frac{y}{b}$ and $\cos\theta = \frac{x}{a}$
$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2\theta + \sin^2\theta = 1.$



5. Vertices are $(0, \pm 2) \Rightarrow a = 2$; $e = \frac{c}{a} \Rightarrow 0.5 = \frac{c}{2} \Rightarrow c = 1 \Rightarrow$ foci are $(0, \pm 1)$

6. Let the center of the ellipse be $(x, 0)$; directrix $x = 2$, focus $(4, 0)$, and $e = \frac{2}{3} \Rightarrow \frac{a}{e} - c = 2 \Rightarrow \frac{a}{e} = 2 + c$ $\Rightarrow a = \frac{2}{3}(2 + c)$. Also $c = ae = \frac{2}{3}a \Rightarrow a = \frac{2}{3}\left(2 + \frac{2}{3}a\right) \Rightarrow a = \frac{4}{3} + \frac{4}{9}a \Rightarrow \frac{5}{9}a = \frac{4}{3} \Rightarrow a = \frac{12}{5}$; $x - 2 = \frac{a}{e}$ $\Rightarrow x - 2 = \left(\frac{12}{5}\right)\left(\frac{3}{2}\right) = \frac{18}{5} \Rightarrow x = \frac{28}{5} \Rightarrow$ the center is $\left(\frac{28}{5}, 0\right)$; $x - 4 = c \Rightarrow c = \frac{28}{5} - 4 = \frac{8}{5}$ so that $c^2 = a^2 - b^2$ $= \left(\frac{12}{5}\right)^2 - \left(\frac{8}{5}\right)^2 = \frac{80}{25}$; therefore the equation is $\frac{\left(x - \frac{28}{5}\right)^2}{\left(\frac{144}{25}\right)} + \frac{y^2}{\left(\frac{80}{25}\right)} = 1$ or $\frac{25\left(x - \frac{28}{5}\right)^2}{144} + \frac{5y^2}{16} = 1$

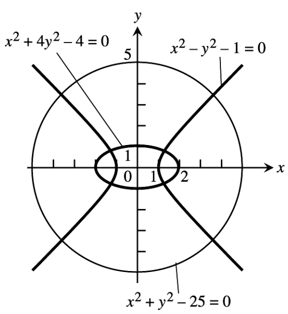
7. Let the center of the hyperbola be $(0, y)$.
   (a) Directrix $y = -1$, focus $(0, -7)$ and $e = 2 \Rightarrow c - \frac{a}{e} = 6 \Rightarrow \frac{a}{e} = c - 6 \Rightarrow a = 2c - 12$. Also $c = ae = 2a$ $\Rightarrow a = 2(2a) - 12 \Rightarrow a = 4 \Rightarrow c = 8$; $y - (-1) = \frac{a}{e} = \frac{4}{2} = 2 \Rightarrow y = 1 \Rightarrow$ the center is $(0, 1)$; $c^2 = a^2 + b^2$ $\Rightarrow b^2 = c^2 - a^2 = 64 - 16 = 48$; therefore the equation is $\frac{(y-1)^2}{16} - \frac{x^2}{48} = 1$
   (b) $e = 5 \Rightarrow c - \frac{a}{e} = 6 \Rightarrow \frac{a}{e} = c - 6 \Rightarrow a = 5c - 30$. Also, $c = ae = 5a \Rightarrow a = 5(5a) - 30 \Rightarrow 24a = 30 \Rightarrow a = \frac{5}{4}$ $\Rightarrow c = \frac{25}{4}$; $y - (-1) = \frac{a}{e} = \frac{\left(\frac{5}{4}\right)}{5} = \frac{1}{4} \Rightarrow y = -\frac{3}{4} \Rightarrow$ the center is $\left(0, -\frac{3}{4}\right)$; $c^2 = a^2 + b^2 \Rightarrow b^2 = c^2 - a^2$ $= \frac{625}{16} - \frac{25}{16} = \frac{75}{2}$; therefore the equation is $\frac{\left(y + \frac{3}{4}\right)^2}{\left(\frac{25}{16}\right)} - \frac{x^2}{\left(\frac{75}{2}\right)} = 1$ or $\frac{16\left(y + \frac{3}{4}\right)^2}{25} - \frac{2x^2}{75} = 1$

8. The center is $(0, 0)$ and $c = 2 \Rightarrow 4 = a^2 + b^2 \Rightarrow b^2 = 4 - a^2$. The equation is $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \Rightarrow \frac{49}{a^2} - \frac{144}{b^2} = 1$ $\Rightarrow \frac{49}{a^2} - \frac{144}{(4-a^2)} = 1 \Rightarrow 49\left(4 - a^2\right) - 144a^2 = a^2\left(4 - a^2\right) \Rightarrow 196 - 49a^2 - 144a^2 = 4a^2 - a^4 \Rightarrow a^4 - 197a^2 + 196$ $= 0 \Rightarrow \left(a^2 - 196\right)\left(a^2 - 1\right) = 0 \Rightarrow a = 14$ or $a = 1$; $a = 14 \Rightarrow b^2 = 4 - (14)^2 < 0$ which is impossible; $a = 1$ $\Rightarrow b^2 = 4 - 1 = 3$; therefore the equation is $y^2 - \frac{x^2}{3} = 1$

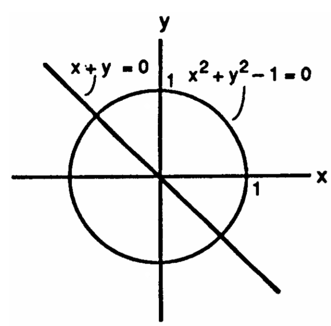
9. $b^2x^2 + a^2y^2 = a^2b^2 \Rightarrow \frac{dy}{dx} = -\frac{b^2x}{a^2y}$; at (x_1, y_1) the tangent line is $y - y_1 = \left(-\frac{b^2x_1}{a^2y_1}\right)(x - x_1)$ $\Rightarrow a^2yy_1 + b^2xx_1 = b^2x_1^2 + a^2y_1^2 = a^2b^2 \Rightarrow b^2xx_1 + a^2yy_1 - a^2b^2 = 0$

10. $b^2x^2 - a^2y^2 = a^2b^2 \Rightarrow \frac{dy}{dx} = \frac{b^2x}{a^2y}$; at (x_1, y_1) the tangent line is $y - y_1 = \left(\frac{b^2x_1}{a^2y_1}\right)(x - x_1)$ $\Rightarrow b^2xx_1 - a^2yy_1 = b^2x_1^2 - a^2y_1^2 = a^2b^2 \Rightarrow b^2xx_1 - a^2yy_1 - a^2b^2 = 0$

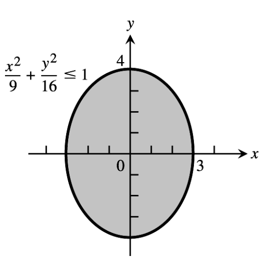
11.



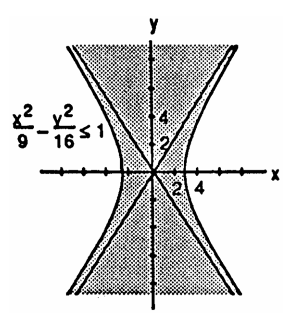
12.



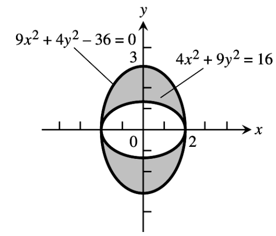
13.



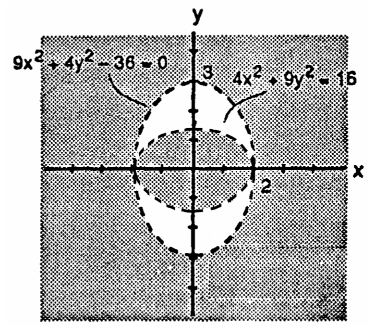
14.



15. $(9x^2 + 4y^2 - 36)(4x^2 + 9y^2 - 16) \le 0$
$\Rightarrow\ 9x^2 + 4y^2 - 36 \le 0$ and $4x^2 + 9y^2 - 16 \ge 0$
or $9x^2 + 4y^2 - 36 \ge 0$ and $4x^2 + 9y^2 - 16 \le 0$

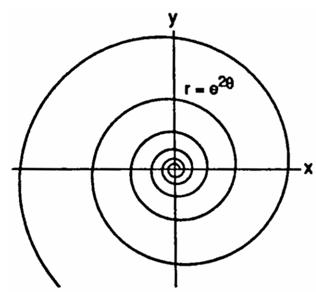


16. $(9x^2 + 4y^2 - 36)(4x^2 + 9y^2 - 16) > 0$, which is the complement of the set in Exercise 15



17. (a) $x = e^{2t} \cos t$ and $y = e^{2t} \sin t\ \Rightarrow\ x^2 + y^2 = e^{4t} \cos^2 t + e^{4t} \sin^2 t = e^{4t}$. Also $\frac{y}{x} = \frac{e^{2t} \sin t}{e^{2t} \cos t} = \tan t$
$\Rightarrow\ t = \tan^{-1}\left(\frac{y}{x}\right)\ \Rightarrow\ x^2 + y^2 = e^{4 \tan^{-1}(y/x)}$ is the Cartesian equation. Since $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$, the polar equation is $r^2 = e^{4\theta}$ or $r = e^{2\theta}$ for $r > 0$

(b) $ds^2 = r^2\, d\theta^2 + dr^2$; $r = e^{2\theta}\ \Rightarrow\ dr = 2e^{2\theta}\, d\theta$
$\Rightarrow\ ds^2 = r^2\, d\theta^2 + \left(2e^{2\theta}\, d\theta\right)^2 = \left(e^{2\theta}\right)^2 d\theta^2 + 4e^{4\theta}\, d\theta^2$
$= 5e^{4\theta}\, d\theta^2\ \Rightarrow\ ds = \sqrt{5}\, e^{2\theta}\, d\theta \Rightarrow L = \int_0^{2\pi} \sqrt{5}\, e^{2\theta}\, d\theta$
$= \left[\frac{\sqrt{5}\, e^{2\theta}}{2}\right]_0^{2\pi} = \frac{\sqrt{5}}{2}\left(e^{4\pi} - 1\right)$



18. $r = 2 \sin^3\left(\frac{\theta}{3}\right)\ \Rightarrow\ dr = 2 \sin^2\left(\frac{\theta}{3}\right) \cos\left(\frac{\theta}{3}\right) d\theta\ \Rightarrow\ ds^2 = r^2\, d\theta^2 + dr^2 = \left[2 \sin^3\left(\frac{\theta}{3}\right)\right]^2 d\theta^2 + \left[2 \sin^2\left(\frac{\theta}{3}\right) \cos\left(\frac{\theta}{3}\right) d\theta\right]^2$
$= 4 \sin^6\left(\frac{\theta}{3}\right) d\theta^2 + 4 \sin^4\left(\frac{\theta}{3}\right) \cos^2\left(\frac{\theta}{3}\right) d\theta^2 = \left[4 \sin^4\left(\frac{\theta}{3}\right)\right]\left[\sin^2\left(\frac{\theta}{3}\right) + \cos^2\left(\frac{\theta}{3}\right)\right] d\theta^2 = 4 \sin^4\left(\frac{\theta}{3}\right) d\theta^2$
$\Rightarrow\ ds = 2 \sin^2\left(\frac{\theta}{3}\right) d\theta$. Then $L = \int_0^{3\pi} 2 \sin^2\left(\frac{\theta}{3}\right) d\theta = \int_0^{3\pi} \left[1 - \cos\left(\frac{2\theta}{3}\right)\right] d\theta = \left[\theta - \frac{3}{2} \sin\left(\frac{2\theta}{3}\right)\right]_0^{3\pi} = 3\pi$

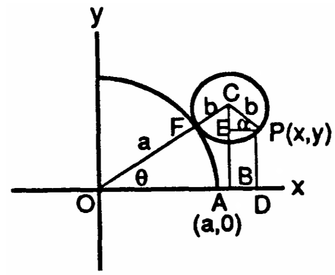
19. $e = 2$ and $r \cos\theta = 2\ \Rightarrow\ x = 2$ is the directrix $\Rightarrow\ k = 2$; the conic is a hyperbola with $r = \frac{ke}{1 + e \cos\theta}$
$\Rightarrow\ r = \frac{(2)(2)}{1 + 2\cos\theta} = \frac{4}{1 + 2\cos\theta}$

20. $e = 1$ and $r \cos\theta = -4\ \Rightarrow\ x = -4$ is the directrix $\Rightarrow\ k = 4$; the conic is a parabola with $r = \frac{ke}{1 - e \cos\theta}$
$\Rightarrow\ r = \frac{(4)(1)}{1 - \cos\theta} = \frac{4}{1 - \cos\theta}$

21. $e = \frac{1}{2}$ and $r \sin\theta = 2\ \Rightarrow\ y = 2$ is the directrix $\Rightarrow\ k = 2$; the conic is an ellipse with $r = \frac{ke}{1 + e \sin\theta}$
$\Rightarrow\ r = \frac{2\left(\frac{1}{2}\right)}{1 + \left(\frac{1}{2}\right) \sin\theta} = \frac{2}{2 + \sin\theta}$

22. $e = \frac{1}{3}$ and $r \sin\theta = -6\ \Rightarrow\ y = -6$ is the directrix $\Rightarrow\ k = 6$; the conic is an ellipse with $r = \frac{ke}{1 - e \sin\theta}$
$\Rightarrow\ r = \frac{6\left(\frac{1}{3}\right)}{1 - \left(\frac{1}{3}\right) \sin\theta} = \frac{6}{3 - \sin\theta}$

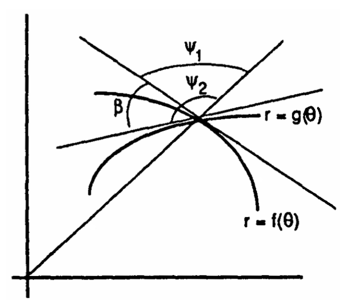
23. Arc PF = Arc AF since each is the distance rolled;
$\angle PCF = \frac{\text{Arc PF}}{b} \Rightarrow \text{Arc PF} = b(\angle PCF)$; $\theta = \frac{\text{Arc AF}}{a}$
$\Rightarrow \text{Arc AF} = a\theta \Rightarrow a\theta = b(\angle PCF) \Rightarrow \angle PCF = \left(\frac{a}{b}\right)\theta$;
$\angle OCB = \frac{\pi}{2} - \theta$ and $\angle OCB = \angle PCF - \angle PCE$
$= \angle PCF - \left(\frac{\pi}{2} - \alpha\right) = \left(\frac{a}{b}\right)\theta - \left(\frac{\pi}{2} - \alpha\right) \Rightarrow \frac{\pi}{2} - \theta$
$= \left(\frac{a}{b}\right)\theta - \left(\frac{\pi}{2} - \alpha\right) \Rightarrow \frac{\pi}{2} - \theta = \left(\frac{a}{b}\right)\theta - \frac{\pi}{2} + \alpha$
$\Rightarrow \alpha = \pi - \theta - \left(\frac{a}{b}\right)\theta \Rightarrow \alpha = \pi - \left(\frac{a+b}{b}\right)\theta$.



Now $x = OB + BD = OB + EP = (a+b)\cos\theta + b\cos\alpha = (a+b)\cos\theta + b\cos\left(\pi - \left(\frac{a+b}{b}\right)\theta\right)$
$= (a+b)\cos\theta + b\cos\pi\cos\left(\left(\frac{a+b}{b}\right)\theta\right) + b\sin\pi\sin\left(\left(\frac{a+b}{b}\right)\theta\right) = (a+b)\cos\theta - b\cos\left(\left(\frac{a+b}{b}\right)\theta\right)$ and
$y = PD = CB - CE = (a+b)\sin\theta - b\sin\alpha = (a+b)\sin\theta - b\sin\left(\left(\frac{a+b}{b}\right)\theta\right)$
$= (a+b)\sin\theta - b\sin\pi\cos\left(\left(\frac{a+b}{b}\right)\theta\right) + b\cos\pi\sin\left(\left(\frac{a+b}{b}\right)\theta\right) = (a+b)\sin\theta - b\sin\left(\left(\frac{a+b}{b}\right)\theta\right)$;
therefore $x = (a+b)\cos\theta - b\cos\left(\left(\frac{a+b}{b}\right)\theta\right)$ and $y = (a+b)\sin\theta - b\sin\left(\left(\frac{a+b}{b}\right)\theta\right)$

24. $x = a(t - \sin t) \Rightarrow \frac{dx}{dt} = a(1 - \cos t)$ and let $\delta = 1 \Rightarrow dm = dA = y\,dx = y\left(\frac{dx}{dt}\right)dt$
$= a(1-\cos t)\,a(1-\cos t)\,dt = a^2(1-\cos t)^2\,dt$; then $A = \int_0^{2\pi} a^2(1-\cos t)^2\,dt$
$= a^2\int_0^{2\pi}(1 - 2\cos t + \cos^2 t)\,dt = a^2\int_0^{2\pi}\left(1 - 2\cos t + \frac{1}{2} + \frac{1}{2}\cos 2t\right)dt = a^2\left[\frac{3}{2}t - 2\sin t + \frac{\sin 2t}{4}\right]_0^{2\pi}$
$= 3\pi a^2$; $\tilde{x} = x = a(t - \sin t)$ and $\tilde{y} = \frac{1}{2}y = \frac{1}{2}a(1-\cos t) \Rightarrow M_x = \int \tilde{y}\,dm = \int \tilde{y}\,\delta\,dA$
$= \int_0^{2\pi}\frac{1}{2}a(1-\cos t)\,a^2(1-\cos t)^2\,dt = \frac{1}{2}a^3\int_0^{2\pi}(1-\cos t)^3\,dt = \frac{a^3}{2}\int_0^{2\pi}(1 - 3\cos t + 3\cos^2 t - \cos^3 t)\,dt$
$= \frac{a^3}{2}\int_0^{2\pi}\left[1 - 3\cos t + \frac{3}{2} + \frac{3\cos 2t}{2} - (1-\sin^2 t)(\cos t)\right]dt = \frac{a^3}{2}\left[\frac{5}{2}t - 3\sin t + \frac{3\sin 2t}{4} - \sin t + \frac{\sin^3 t}{3}\right]_0^{2\pi}$
$= \frac{5\pi a^3}{2}$. Therefore $\overline{y} = \frac{M_x}{M} = \frac{\left(\frac{5\pi a^3}{2}\right)}{3\pi a^2} = \frac{5}{6}a$. Also, $M_y = \int \tilde{x}\,dm = \int \tilde{x}\,\delta\,dA$
$= \int_0^{2\pi} a(t - \sin t)\,a^2(1-\cos t)^2\,dt = a^3\int_0^{2\pi}(t - 2t\cos t + t\cos^2 t - \sin t + 2\sin t\cos t - \sin t\cos^2 t)\,dt$
$= a^3\left[\frac{t^2}{2} - 2\cos t - 2t\sin t + \frac{1}{4}t^2 + \frac{1}{8}\cos 2t + \frac{t}{4}\sin 2t + \cos t + \sin^2 t + \frac{\cos^3 t}{3}\right]_0^{2\pi} = 3\pi^2 a^3$. Thus
$\overline{x} = \frac{M_y}{M} = \frac{3\pi^2 a^3}{3\pi a^2} = \pi a \Rightarrow \left(\pi a, \frac{5}{6}a\right)$ is the center of mass.

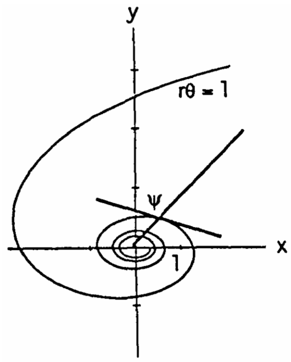
25. $\beta = \psi_2 - \psi_1 \Rightarrow \tan\beta = \tan(\psi_2 - \psi_1) = \frac{\tan\psi_2 - \tan\psi_1}{1 + \tan\psi_2\tan\psi_1}$;
the curves will be orthogonal when $\tan\beta$ is undefined, or
when $\tan\psi_2 = \frac{-1}{\tan\psi_1} \Rightarrow \frac{r}{g'(\theta)} = \frac{-1}{\left[\frac{r}{f'(\theta)}\right]}$
$\Rightarrow r^2 = -f'(\theta)\,g'(\theta)$



26. $r = \sin^4\left(\frac{\theta}{4}\right) \Rightarrow \frac{dr}{d\theta} = \sin^3\left(\frac{\theta}{4}\right)\cos\left(\frac{\theta}{4}\right) \Rightarrow \tan\psi = \frac{\sin^4\left(\frac{\theta}{4}\right)}{\sin^3\left(\frac{\theta}{4}\right)\cos\left(\frac{\theta}{4}\right)} = \tan\left(\frac{\theta}{4}\right)$

27. $r = 2a\sin 3\theta \Rightarrow \frac{dr}{d\theta} = 6a\cos 3\theta \Rightarrow \tan\psi = \frac{r}{\left(\frac{dr}{d\theta}\right)} = \frac{2a\sin 3\theta}{6a\cos 3\theta} = \frac{1}{3}\tan 3\theta$; when $\theta = \frac{\pi}{6}$, $\tan\psi = \frac{1}{3}\tan\frac{\pi}{2} \Rightarrow \psi = \frac{\pi}{2}$

28. (a)



(b) $r\theta = 1 \Rightarrow r = \theta^{-1} \Rightarrow \frac{dr}{d\theta} = -\theta^{-2} \Rightarrow \tan \psi|_{\theta=1}$
$= \frac{\theta^{-1}}{-\theta^{-2}} = -\theta \Rightarrow \lim_{\theta \to \infty} \tan \psi = -\infty$
$\Rightarrow \psi \to \frac{\pi}{2}$ from the right as the spiral winds in around the origin.

29. $\tan \psi_1 = \frac{\sqrt{3} \cos \theta}{-\sqrt{3} \sin \theta} = -\cot \theta$ is $-\frac{1}{\sqrt{3}}$ at $\theta = \frac{\pi}{3}$; $\tan \psi_2 = \frac{\sin \theta}{\cos \theta} = \tan \theta$ is $\sqrt{3}$ at $\theta = \frac{\pi}{3}$; since the product of these slopes is -1, the tangents are perpendicular

30. $\tan \psi = \frac{r}{\left(\frac{dr}{d\theta}\right)} = \frac{a(1 - \cos \theta)}{a \sin \theta}$ is 1 at $\theta = \frac{\pi}{2} \Rightarrow \psi = \frac{\pi}{4}$

NOTES: